The Voronoi formula and double Dirichlet series

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September 19, 2015

Abstract

We prove a Voronoi formula for coefficients of a large class of L-functions including Maass cusp forms, Rankin-Selberg convolutions, and certain isobaric sums. Our proof is based on the functional equations of L-functions twisted by Dirichlet characters and does not directly depend on automorphy. Hence it has wider application than previous proofs. The key ingredient is the construction of a double Dirichlet series.

MSC: 11F30 (Primary), 11F68, 11L05

Keywords: Voronoi formula, automorphic form, Maass form, multiple Dirichlet series, Gauss sum, Kloosterman sum, Rankin-Selberg L-function

1 Introduction

A Voronoi formula is an identity involving Fourier coefficients of automorphic forms, with the coefficients twisted by additive characters on either side. A history of the Voronoi formula can be found in [MS04]. Since its introduction in [MS06], the Voronoi formula on GL(3) of Miller and Schmid has become a standard tool in the study of L-functions arising from GL(3), and has found important applications such as [BB], [BKY13], [Kha12], [Li09], [Li11], [LY12], [Mil06], [Mun13] and [Mun15]. As of yet the general GL(N) formula has had fewer applications, a notable one being [KR14].

The first proof of a Voronoi formula on GL(3) was found by Miller and Schmid in [MS06] using the theory of automorphic distributions. Later, a Voronoi formula was established for GL(N) with $N \geq 4$ in [GL06], [GL08], and [MS11], with [MS11] being more general and earlier than [GL08] (see the addendum, loc. cit.). Goldfeld and Li's proof [GL08] is more akin to the classical proof in GL(2) (see [Goo81]), obtaining the associated Dirichlet series through a shifted "vertical" period integral and making use of automorphy. A very general and adelic version was established by Ichino and Templier in [IT13], allowing ramifications and applications to number fields. Another direction of generalization with more complicated additive twists on either side has been considered in an unpublished work of Li and Miller and in [Zho].

In this article, we prove a Voronoi formula for a large class of automorphic objects or L-functions, including cusp forms for SL($N, \mathbb{Z}$), Rankin-Selberg convolutions, and certain non-cuspidal forms. Previous works ([MS11], [GL08], [IT13]) do not offer a Voronoi formula for Rankin-Selberg convolutions or non-cuspidal forms. Even for Maass cusp forms, our new proof is shorter than any previous one, and uses a completely different set of techniques.

Let us briefly summarize our method of proof. We first reduce the statement of Voronoi formula to a formula involving Gauss sums of Dirichlet characters. We construct a complex function of two variables and write it as double Dirichlet series in two different ways by applying a functional equation. Using the uniqueness theorem of Dirichlet series, we get an identity between coefficients of these two double Dirichlet series. This leads us to the Voronoi formula with Gauss sums.

One of our key steps in obtaining the Voronoi formula is the use of functional equations of L-functions twisted by Dirichlet characters. The relationship between the Voronoi formulas and the functional equations of these L-functions is known from previous works, such as [DI90], Section 4 of [GL06], [BK15] and [Zho]. Miller-Schmid derived the functional equation of L-functions twisted by a Dirichlet character of prime conductor from the Voronoi formula in Section 6 of [MS06]. However there is a combinatorial difficulty in reversing this process, i.e., obtaining additive twists of general non-prime conductors from multiplicative ones, which was mentioned in both [MS06, p. 430] and [IT13, p. 68]. The method presented here is able to overcome this difficulty by discovering an interlocking structure among a family of Voronoi formulas with different conductors.

Our proof of the Voronoi formula is complete for additive twists of all conductors, prime or not, and unlike [GL06], [GL08],[IT13], [MS06], or [MS11], does not depend directly on automorphy of the cusp forms. This fact allows us to apply our theorem to many conjectural Langlands functorial transfers. For example, the Rankin-Selberg convolutions (also called functorial products) for $\mathrm{GL}(m) \times \mathrm{GL}(n)$ are not yet known to be automorphic on $\mathrm{GL}(m \times n)$ in general. Yet we know the functional equations of $\mathrm{GL}(m) \times \mathrm{GL}(n)$ L-functions twisted by Dirichlet characters. Thus, our proof provides a Voronoi formula for the Rankin-Selberg convolutions on $\mathrm{GL}(m) \times \mathrm{GL}(n)$ (see Example 1.6). Voronoi formulas for these functorial cases are unavailable from [GL08], [MS11] or [IT13]. In Theorem 1.3 we reformulate our Voronoi formula like the classical converse theorem of Weil, i.e., assuming every L-function twisted by Dirichlet character is entire, has an Euler product (or satisfies Hecke relations), and satisfies the precise functional equations, then the Voronoi formula as in Theorem 1.1 is valid. We do not have to assume it is a standard L-function coming from a cusp form.

By Theorem 1.3 we obtain a Voronoi formula for certain non-cuspidal forms, such as isobaric sums (see Example 1.7). This is not readily available from any previous work but it is believed (see [MS11] p. 176) that one may derive a formula by using formulas on smaller groups through a possibly complicated procedure. Such complication does not occur in our method because we work directly with L-functions.

We first state the main results for Maass cusp forms. Denote $e(x) := \exp(2\pi i x)$. Let $a, n \in \mathbb{Z}$, $c \in \mathbb{N}$ and let

$$\mathbf{q} = (q_1, q_2, \ldots, q_{N-2}) \qquad \text{and} \qquad \mathbf{d} = (d_1, d_2, \ldots, d_{N-2}),$$

be tuples of positive integers satisfying the divisibility conditions

$$d_1 | q_1 c, \quad d_2 \left| \frac{q_1 q_2 c}{d_1}, \qquad \cdots, \qquad d_{N-2} \right| \frac{q_1 \cdots q_{N-2} c}{d_1 \cdots d_{N-3}}. \tag{1}$$

Define the hyper-Kloosterman sum as

$$\begin{aligned}\mathrm{Kl}(a, n, c; \mathbf{q}, \mathbf{d}) = & \sideset{}{^*}\sum_{x_1 (\bmod \frac{q_1 c}{d_1})} \ \sideset{}{^*}\sum_{x_2 (\bmod \frac{q_1 q_2 c}{d_1 d_2})} \cdots \sideset{}{^*}\sum_{x_{N-2} (\bmod \frac{q_1 \cdots q_{N-2} c}{d_1 \cdots d_{N-2}})} \\ & \times e\left(\frac{d_1 x_1 a}{c} + \frac{d_2 x_2 \overline{x_1}}{\frac{q_1 c}{d_1}} + \cdots + \frac{d_{N-2} x_{N-2} \overline{x_{N-3}}}{\frac{q_1 \cdots q_{N-3} c}{d_1 \cdots d_{N-3}}} + \frac{n \overline{x_{N-2}}}{\frac{q_1 \cdots q_{N-2} c}{d_1 \cdots d_{N-2}}} \right),\end{aligned}$$

where $\sum^*$ indicates that the summation is over reduced residue classes, and $\overline{x_i}$ denotes the multiplicative inverse of x_i modulo $q_1 \cdots q_i c / d_1 \cdots d_i$. When $N = 3$, $\mathrm{Kl}(a, n, c; q_1, d_1)$ becomes the classical Kloosterman sum $S(aq_1, n; cq_1/d_1)$.

Let F be a Hecke-Maass cusp form for $\mathrm{SL}(N, \mathbb{Z})$ with the spectral parameters $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$. Let $A(*, \ldots, *)$ be the Fourier-Whittaker coefficients of F normalized as $A(1, \ldots, 1) = 1$. We refer to [Gol06] for the definitions and the basic results of Maass forms for $\mathrm{SL}(N, \mathbb{Z})$. The Fourier coefficients satisfy the Hecke relations

$$A(m_1 m_1', \cdots, m_{N-1} m_{N-1}') = A(m_1, \ldots, m_{N-1}) A(m_1', \ldots, m_{N-1}') \tag{2}$$

where $(m_1 \cdots m_{N-1}, m_1' \cdots m_{N-1}') = 1$ is satisfied,

$$A(1, \ldots, 1, n) A(m_{N-1}, \ldots, m_1) = \sum_{\substack{d_0 \ldots d_{N-1} = n \\ d_1 | m_1, \ldots, d_{N-1} | m_{N-1}}} A\left(\frac{m_{N-1} d_{N-2}}{d_{N-1}}, \ldots, \frac{m_2 d_1}{d_2}, \frac{m_1 d_0}{d_1} \right), \tag{3}$$

and

$$A(n, 1, \ldots, 1) A(m_1, \ldots, m_{N-1}) = \sum_{\substack{d_0 \ldots d_{N-1} = n \\ d_1 | m_1, \ldots, d_{N-1} | m_{N-1}}} A\left(\frac{m_1 d_0}{d_1}, \frac{m_2 d_1}{d_2}, \ldots, \frac{m_{N-1} d_{N-2}}{d_{N-1}} \right). \tag{4}$$

The dual Maass form of F is denoted by $\widetilde{F}$. Let $B(*,\ldots,*)$ be the Fourier-Whittaker coefficients of $\widetilde{F}$. These coefficients satisfy

$$B(m_1,\ldots,m_{N-1}) = A(m_{N-1},\ldots,m_1). \tag{5}$$

Define the ratio of Gamma factors

$$G_\pm(s) := i^{-N\delta}\pi^{-N(1/2-s)}\prod_{j=1}^{N}\Gamma\left(\frac{\delta+1-s-\overline{\lambda_j}}{2}\right)\Gamma\left(\frac{\delta+s-\lambda_j}{2}\right)^{-1}, \tag{6}$$

where for even Maass forms, we define $\delta=0$ in G_+ and $\delta=1$ in G_- and for odd Maass forms, we define $\delta=1$ in G_+ and $\delta=0$ in G_-. We refer to Section 9.2 of [Gol06] for the definition of even and odd Maass forms.

Theorem 1.1 (Voronoi formula on GL(N) of Miller-Schmid [MS11])**.** *Let F be a Hecke-Maass cusp form with coefficients $A(*,\ldots,*)$, and $G_\pm$ a ratio of Gamma factors as in (6). Let $c>0$ be an integer and let a be any integer with $(a,c)=1$. Denote by $\overline{a}$ the multiplicative inverse of a modulo c. Let the additively twisted Dirichlet series be given as*

$$L_{\mathbf{q}}(s,F,a/c) = \sum_{n=1}^{\infty}\frac{A(q_{N-2},\ldots,q_1,n)}{n^s}e\left(\frac{\overline{a}n}{c}\right). \tag{7}$$

This Dirichlet series has an analytic continuation to all $s\in\mathbb{C}$ and satisfies the functional equation

$$\begin{aligned}
L_{\mathbf{q}}(s,F,a/c) =& \frac{G_+(s)+G_-(s)}{2}\sum_{d_1|q_1c}\sum_{d_2|\frac{q_1q_2c}{d_1}}\cdots\sum_{d_{N-2}|\frac{q_1\cdots q_{N-2}c}{d_1\cdots d_{N-3}}}\\
&\times\sum_{n=1}^{\infty}\frac{A(n,d_{N-2},\ldots,d_2,d_1)\,\mathrm{Kl}(a,n,c;\mathbf{q},\mathbf{d})}{n^{1-s}c^{Ns-1}d_1d_2\cdots d_{N-2}}\frac{d_1^{(N-1)s}d_2^{(N-2)s}\cdots d_{N-2}^{2s}}{q_1^{(N-2)s}q_2^{(N-3)s}\cdots q_{N-2}^{s}}\\
&+\frac{G_+(s)-G_-(s)}{2}\sum_{d_1|q_1c}\sum_{d_2|\frac{q_1q_2c}{d_1}}\cdots\sum_{d_{N-2}|\frac{q_1\cdots q_{N-2}c}{d_1\cdots d_{N-3}}}\\
&\times\sum_{n=1}^{\infty}\frac{A(n,d_{N-2},\ldots,d_2,d_1)\,\mathrm{Kl}(a,-n,c;\mathbf{q},\mathbf{d})}{n^{1-s}c^{Ns-1}d_1d_2\cdots d_{N-2}}\frac{d_1^{(N-1)s}d_2^{(N-2)s}\cdots d_{N-2}^{2s}}{q_1^{(N-2)s}q_2^{(N-3)s}\cdots q_{N-2}^{s}},
\end{aligned} \tag{8}$$

in the region of convergence of the right hand side.

The traditional Voronoi formula, involving weight functions instead of Dirichlet series, is obtained after taking an inverse Mellin transform against a suitable test function.

Choose a Dirichlet character χ modulo c, which is not necessarily primitive, multiply both sides of (8) by $\chi(a)$, and sum this equality over the reduced residue system modulo c. We obtain the following the Voronoi formula with Gauss sums. In Section 5.3 we show through elementary finite arithmetic that the formulas (8) and (11) are equivalent.

Theorem 1.2 (Voronoi formula with Gauss sums)**.** *Let χ be a Dirichlet character modulo c, induced from the primitive character χ^* modulo c^* with $c^*|c$. Define for $\mathbf{q}=(q_1,\ldots,q_{N-2})$ a tuple of positive integers*

$$H(\mathbf{q},c,\chi^*,s) = \sum_{n=1}^{\infty}\frac{A(q_{N-2},\ldots,q_1,n)g(\overline{\chi^*},c,n)}{n^s(c/c^*)^{1-2s}}, \tag{9}$$

and

$$\begin{aligned}
G(\mathbf{q},c,\chi^*,s) =& \frac{G(s)}{c^{Ns-1}(c/c^*)^{1-2s}}\sum_{d_1c^*|q_1c}\sum_{d_2c^*|\frac{q_1q_2c}{d_1}}\cdots\sum_{d_{N-2}c^*|\frac{q_1\cdots q_{N-2}c}{d_1\cdots d_{N-3}}}\\
&\times\sum_{n=1}^{\infty}\frac{A(n,d_{N-2},\ldots,d_1)}{n^{1-s}d_1d_2\cdots d_{N-2}}\frac{d_1^{(N-1)s}d_2^{(N-2)s}\cdots d_{N-2}^{2s}}{q_1^{(N-2)s}q_2^{(N-3)s}\cdots q_{N-2}^{s}}\\
&\times g(\chi^*,c,d_1)g(\chi^*,\tfrac{q_1c}{d_1},d_2)\cdots g(\chi^*,\tfrac{q_1\cdots q_{N-3}c}{d_1\cdots d_{N-3}},d_{N-2})g(\chi^*,\tfrac{q_1\cdots q_{N-2}c}{d_1\cdots d_{N-2}},n),
\end{aligned} \tag{10}$$

where G equals G_+ or G_- depending on whether $\chi^*(-1)$ is 1 or -1, and $g(\chi^*, \ell c^*, *)$ is the Gauss sum of the induced character modulo ℓc^* from χ^*, which is defined in Definition 2.1. Both functions have analytic continuation to all $s \in \mathbb{C}$, and the equality

$$H(\mathbf{q}, c, \chi^*, s) = G(\mathbf{q}, c, \chi^*, s) \tag{11}$$

is satisfied.

In proving (11), we define

$$Z(s, w) = \frac{L_{\mathbf{q}}(2w - s, F)L(s, F \times \chi^*)}{L(2w - 2s + 1, \overline{\chi^*})}, \tag{12}$$

where $\mathbf{q} = (q_1, \dots, q_{N-2})$ is a tuple of positive integers, and the function $L_{\mathbf{q}}(s, F)$ is given as the Dirichlet series

$$L_{\mathbf{q}}(s, F) = \sum_{n=1}^{\infty} \frac{A(q_{N-2}, \dots, q_1, n)}{n^s},$$

for $\Re(s) \gg 1$. We express $Z(s, w)$ as a double Dirichlet series in two different ways. In one region of convergence we express the L-functions as Dirichlet series and obtain

$$Z(s, w) = \sum_{n=1}^{\infty} \frac{a_n(s)}{n^{2w}}.$$

On the other hand we apply the functional equation of $L(s, F \times \chi^*)$, replacing s with $1 - s$, and write $Z(s, w)$ as the Dirichlet series

$$Z(s, w) = \sum_{n} \frac{b_n(s)}{n^{2w}}.$$

By the uniqueness of Dirichlet series, we must have $a_n(s) = b_n(s)$. This equality leads us to the Voronoi formula with Gauss sums.

Our proof only uses the Hecke relations about the Fourier coefficients of F and the exact form of the functional equations. The expression of Gamma factors, or the automorphy of F, plays no role. Hence we can formulate our theorem in a style similar to the classical converse theorem of Weil. First let us list the properties of Fourier coefficients that we use in order to state the following theorem.

The Fourier coefficients of F grow moderately, i.e.,

$$A(m_1, \dots, m_{N-1}) \ll (m_1 \dots m_{N-1})^{\sigma} \tag{13}$$

for some $\sigma > 0$. Given a primitive Dirichlet character χ^* modulo c^*, define the twisted L-function

$$L(s, F \times \chi^*) = \sum_{n=1}^{\infty} \frac{A(1, \dots, 1, n)\chi^*(n)}{n^s}, \tag{14}$$

for $\Re(s) > \sigma + 1$. It has analytic continuation to the whole complex plane, and satisfies the functional equation

$$L(s, F \times \chi^*) = \tau(\chi^*)^N c^{*-Ns} G(s) L(1 - s, \widetilde{F} \times \overline{\chi^*}), \tag{15}$$

where $G(s) = G_+(s)$ or $G_-(s)$ depending on whether $\chi^*(-1) = 1$ or -1.

Theorem 1.3. *Let F be a symbol and assume that with F comes numbers $A(m_1, \dots, m_{N-1}) \in \mathbb{C}$ attached to every $(N-1)$-tuple $(m_1, \dots, m_{N-1})$ of natural numbers. Assume $A(1, \dots, 1) = 1$.*

Assume that these "coefficients" $A(*, \dots, *)$ satisfy the aforementioned Hecke relations (2), (3) *and* (4). *Further assume that they grow moderately as in* (13).

Let $\widetilde{F}$ be another symbol whose associated coefficients $B(*, \dots, *) \in \mathbb{C}$ are given as in (5) *and assume that they also satisfy the same properties. Also assume that there are two meromorphic functions $G_+(s)$ and $G_-(s)$ associated to the pair $(F, \widetilde{F})$, so that for a given primitive character χ^*, the function $L(s, F \times \chi^*)$ as defined in* (14) *satisfies the functional equation* (15).

Under all these assumptions, $L_{\mathbf{q}}(s, F, a/c)$ defined as in (7), *has analytic continuation to all $s \in \mathbb{C}$, and satisfies the Voronoi formula* (8).

Equivalently the functions $H(\mathbf{q}, c, \chi^*, s)$ and $G(\mathbf{q}, c, \chi^*, s)$ as defined by the formulas (9) *and* (10) *have analytic continuations to all s and equal each other as in* (11).

Remark 1.4. If we start with an L-series $L(s,F)$ with an Euler product

$$L(s,F)=\sum_{n=1}^{\infty}\frac{A(1,\ldots,1,n)}{n^s}=\prod_p\prod_{i=1}^{N}\left(1-\frac{\alpha_i(p)}{p^s}\right)^{-1}$$

and with $\prod_i \alpha_i(p)=1$ for any p, we can define $A(p^{k_1},\ldots,p^{k_{N-1}})$ by the Casselman-Shalika formula (Proposition 5.1 of [Zho14]) and they are compatible with the Hecke relations. More explicitly, for a prime number p, we define $A(p^{k_1},\ldots,p^{k_{N-1}})=S_{k_{N-1},\ldots,k_1}(\alpha_i(p),\ldots,\alpha_N(p))$ by the work of Shintani where $S_{k_{N-1},\ldots,k_1}(*)$ is the Schur polynomial, which can be found in (2.7) of [KR14]. We extend the definition to all $A(*,\ldots,*)$ multiplicatively by (2). In summary, the "coefficients" $A(*,\ldots,*)$ along with the Hecke relations can be generated by an L-function with an Euler product.

The following examples satisfy the conditions in Theorem 1.3 and hence we have a Voronoi formula for each of them.

Example 1.5 (Automorphic form for $\mathrm{SL}(N,\mathbb{Z})$)**.** Any cuspidal automorphic form for $\mathrm{SL}(N,\mathbb{Z})$ satisfies the conditions in Theorem 1.3. It can have an unramified or ramified component at the archimedean place, because only the exact form of the $G_\pm$ function would change (see [GJ72]). The Hecke-Maass cusp forms considered in Theorem 1.1 are included in this category, and therefore, we prove Theorem 1.3 instead of Theorem 1.1.

Example 1.6 (Rankin-Selberg convolution)**.** Let F_1 be a Hecke-Maass cusp form for $\mathrm{SL}(N_1,\mathbb{Z})$ and let F_2 be a Hecke-Maass cusp form for $\mathrm{SL}(N_2,\mathbb{Z})$. Assume $F_1\neq\widetilde{F}_2$ if $N_1=N_2$. Define $F=F_1\times F_2$ to be the Rankin-Selberg convolution of F_1 and F_2. The work of Jacquet, Piatetski-Shapiro, and Shalika [JPSS83] shows that $L(s,F\times\chi^*)=L(s,(F_1\times\chi^*)\times F_2)$ is holomorphic and satisfies the functional equation (15).

Example 1.7 (Isobaric sum, Eisenstein series)**.** For $i=1,\ldots,k$ let F_i be a Hecke-Maass cusp form for $\mathrm{SL}(N_i,\mathbb{Z})$. Let s_i be complex numbers with $\sum_i N_i s_i=0$. Define the isobaric sum $F=(F_1\times|\cdot|_{\mathbb{A}}^{s_1})\boxplus(F_2\times|\cdot|_{\mathbb{A}}^{s_2})\boxplus\cdots\boxplus(F_k\times|\cdot|_{\mathbb{A}}^{s_k})$, whose L-function is $L(s,F)=\prod_i L(s+s_i,F_i)$. This isobaric sum F is associated with a non-cuspidal automorphic form on $\mathrm{GL}(N)$, an Eisenstein series twisted by Maass forms, where $N=\sum_i N_i$ (see [Gol06, Section 10.5]). The L-function twisted by a character is simply given by $L(s,F\times\chi^*)=\prod_i L(s+s_i,F_i\times\chi^*)$ which satisfies the conditions of Theorem 1.3.

Example 1.8 (Symmetric powers on $\mathrm{GL}(2)$)**.** Let f be a modular form of weight k for $\mathrm{SL}(2,\mathbb{Z})$ and define $F:=\mathrm{Sym}^2 f$. The symmetric square F satisfies the conditions in Theorem 1.3 by the work of Shimura, [Shi75]. Here we do not need to involve automorphy using Gelbart-Jacquet lifting. One may have similar results for higher symmetric powers depending on the recent progress in the theory of Galois representations.

As a last remark, let us explain the construction of the double Dirichlet series $Z(s,w)$ given by (12). This construction originates from the Rankin-Selberg convolution of a cusp form F and an Eisenstein series on $\mathrm{GL}(2)$. The Fourier coefficients of the Eisenstein series $E(z,s,\chi^*)$ can be written as

$$\frac{1}{n^{2s-1}}\frac{\sigma_{2s-1}(n,\chi^*)}{L(2s,\overline{\chi^*})}\quad\text{or}\quad\sum_{\ell=1}^{\infty}\frac{g(\overline{\chi^*},\ell c^*,n)}{(\ell c^*)^{2s}}.$$

Therefore, in the case of F on $\mathrm{GL}(2)$, the Rankin-Selberg integral of F and $E(*,w-s+1/2,\chi^*)$ produces the double Dirichlet series

$$\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{A(n)g(\overline{\chi^*},\ell c^*,n)}{n^s(\ell c^*)^{2w+1-2s}}.$$

A similar expression appears on the left hand side of the Voronoi formula with Gauss sums (9). The Rankin-Selberg convolution of the cusp form F and an Eisenstein series can be written as a product of two copies of standard L-function of F, namely

$$\frac{L(2w-s,F)L(s,F\times\chi^*)}{L(2w-2s+1,\overline{\chi^*})}.$$

Applying the functional equation to only $L(s,F\times\chi^*)$ gives us another expression, which is similar to the right hand side (10) of the Voronoi formula with Gauss sums. Since $L(2w-s,F)$ was not used in this process, we have the freedom to replace $L(2w-s,F)$ by $L_{\mathbf{q}}(2w-s,F)$ in the case of GL(N) and it gives us enough generality to prove the full Voronoi formula (10). In the case of GL(3), this construction is similar to Bump's double Dirichlet series, see [Gol06, Chapter 6.6] or [Bum84, Chapter X].

Here is a brief overview of the article. In Section 3 we prove the Voronoi formula for a modular form of weight k, as well as for Maass cusp forms. In the first proof for modular forms we do this by the method of unfolding an integral containing two Eisenstein series in two different ways. The second proof, for Maass forms, is to take consideration of L-functions directly and is more close to the general case of GL(N).

In Section 2 we establish some notation and record some formulas about Gauss sums to be used later.

In Section 3 we prove the Voronoi formula on GL(2) in two different ways. In Section 3.1 we obtain the functional equation of the formula using unfolding of different Eisenstein series. In Section 3.2 we prove the same formula once again, where the method of proof is more aligned with the general proof. The factorized form of the double Dirichlet series involves an extra Dirichlet L-function in the numerator in GL(2).

In Section 4 we prove the GL(3) Voronoi formula, which is identical *mutatis mutandis* with the general proof for L-functions of degree N, but fewer indices in the main computation of the proof of Theorem 4.8 makes the proof easier to follow and hence we repeat the proof for convenience.

Section 5 is where we start the general proof in earnest, and cognoscenti may read only Sections 2 and 5 for the proof of the Voronoi formula on GL(N).

2 Background on Gauss sums

Here we collect information about the Gauss sums of Dirichlet characters which are not necessarily primitive.

Definition 2.1. Let χ be a Dirichlet character modulo c induced from a primitive Dirichlet character χ^* modulo c^*. Define the divisor function

$$\sigma_s(m,\chi)=\sum_{d|m}\chi(d)d^s.$$

Define the Gauss sum of χ

$$g(\chi^*,c,m)=\sum_{\substack{u \mod c\\(u,c)=1}}\chi(u)e\left(\frac{mu}{c}\right),$$

and the standard Gauss sum for χ^* is given as $\tau(\chi^*)=g(\chi^*,c^*,1)$.

Lemma 2.2 (Gauss sum of non-primitive characters, Lemma 3.1.3.(2) of [Miy06])**.** *Let χ be a character modulo c induced from primitive character χ^* modulo c^*. Then the Gauss sum of χ is given by*

$$g(\chi^*,c,a)=\tau(\chi^*)\sum_{d|\left(a,\frac{c}{c^*}\right)}d\chi^*\left(\frac{c}{c^*d}\right)\overline{\chi^*}\left(\frac{a}{d}\right)\mu\left(\frac{c}{c^*d}\right).$$

Lemma 2.3 ([Has64], p. 424)**.** *Let χ^* be a primitive character modulo c^* and assume $c^*|c$. Then, we have*

$$g(\chi^*,c,a)=\tau(\chi^*)\frac{\phi(c)}{\phi\left(\frac{c}{(c,a)}\right)}\mu\left(\frac{c}{c^*(c,a)}\right)\chi^*\left(\frac{c}{c^*(c,a)}\right)\overline{\chi^*}\left(\frac{a}{(c,a)}\right),$$

if $c^*|c/(a,c)$. Otherwise $g(\chi^*,c,a)$ is zero.

Next lemma is a generalization of a famous formula of Ramanujan,

$$\frac{\sigma_{s-1}(n)}{n^{s-1}}=\zeta(s)\sum_{\ell=1}^{\infty}\frac{c_\ell(n)}{\ell^s},$$

where $c_\ell(n)$ is the Ramanujan sum.

Lemma 2.4. *Define a Dirichlet series*

$$I(s,\chi^*,c^*,m)=\sum_{\ell=1}^{\infty}\frac{g(\chi^*,\ell c^*,m)}{\ell^s},$$

as a generating function for the non-primitive Gauss sums induced from χ^*. It satisfies the identity

$$\tau(\chi^*)\sigma_{s-1}(m,\overline{\chi^*})=m^{s-1}I(s,\chi^*,c^*,m)L(s,\chi^*).$$

Proof. Expanding the both sides of $\tau(\chi^*)\sigma_{s-1}(m,\overline{\chi^*})L(s,\chi^*)^{-1}=m^{s-1}I(s,\chi^*,c^*,m)$ gives Lemma 2.2 coefficientwise. ☐

Lemma 2.5. *For any two positive integers n and m, and a primitive Dirichlet character χ^* modulo c^*, we have*

$$\sum_{\ell d=n}\chi^*(d)g(\chi^*,\ell c^*,m)=\begin{cases}\tau(\chi^*)\overline{\chi^*(m/n)}n, & \text{if } n|m,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. We start with the formula,

$$\frac{\tau(\chi^*)\sigma_{s-1}(m,\overline{\chi^*})}{m^{s-1}}=I(s,\chi^*,c^*,m)L(s,\chi^*).$$

Both sides are Dirichlet series and we equate coefficients. The left hand side is given as

$$\tau(\chi^*)\sum_{e|m}\frac{\overline{\chi^*(m/e)}e}{e^s},$$

whereas the right hand side is

$$\sum_{\ell=1}^{\infty}\frac{g(\chi^*,\ell c^*,m)}{\ell^s}\sum_{d=1}^{\infty}\frac{\chi^*(d)}{d^s}=\sum_{n=1}^{\infty}\frac{\sum_{d\ell=n}\chi^*(d)g(\chi^*,\ell c^*,m)}{n^s}.$$ ☐

3 Voronoi Formula on GL(2)

3.1 The Rankin-Selberg Integral in GL_2

One of the ways to obtain the Rankin-Selberg convolution of our chosen automorphic form with an Eisenstein series is to consider a Rankin-Selberg integral with another Eisenstein series. Let us do that in the case of a holomorphic, weight k, modular form f on the full modular group. Let us further assume that k is even.

Let χ be any Dirichlet character modulo N and for $\Re(s)>1$ define the Eisenstein series of level N, nebentypus χ and weight k by

$$E^{(N)}(z,s,\chi,k)=\sum_{\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_0(N)}\overline{\chi(d)}\Im(\gamma z)^s\frac{|j(\gamma,z)|^k}{j(\gamma,z)^k},$$

with $j(\gamma,z)=(cz+d)$ the j-cocyle. The Eisenstein series can be analytically continued to all $s\in\mathbb{C}$.

This Eisenstein series has the Fourier expansion

$$\begin{aligned}E^{(N)}(z,s,\chi,k)=&y^s+\delta_{\chi,\chi_0}y^{1-s}i^{-k}\sum_{\substack{c>0\\c\equiv0\mod N}}\frac{\phi(c)}{c^{2s}}2\pi2^{1-2s}\frac{\Gamma(2s-1)}{\Gamma(s+\frac{k}{2})\Gamma(s-\frac{k}{2})}\\&+\sum_{n\neq0}i^{-k}\sum_{\substack{c>0\\c\equiv0\mod N}}\frac{g(\bar\chi,c,n)}{c^{2s}}\frac{\pi^sn^{s-1}}{\Gamma(s+\mathrm{sign}(n)\frac{k}{2})}W_{\mathrm{sign}(n)\frac{k}{2},\frac12-s}(4\pi|n|y).\end{aligned}$$

For weight 0, this simplifies as,

$$E^{(N)}(z,s,\chi)=y^s+\delta_{\chi,\chi_0}y^{1-s}\sum_{\substack{c>0\\ c\equiv 0 \mod N}}^{\infty}\frac{\phi(Nc)}{(Nc)^{2s}}$$
$$+\sum_{n\neq 0}\left(\sum_{\substack{c>0\\ c\equiv 0 \mod N}}\frac{g(\bar\chi,c,n)}{c^{2s}}\right)|n|^{s-\frac12}\frac{\sqrt{y}K_{s-\frac12}(2\pi|n|y)}{\pi^{-s}\Gamma(s)}.$$

Here δ_{χ,χ_0} is the Kronecker symbol for χ being the trivial character modulo N, and ϕ is the Euler's phi function.

The classical Voronoi transformation formula is equivalent to a functional equation of the additively twisted Dirichlet series,

$$L(w,f,u/N)=\sum_{n=1}^{\infty}\frac{\lambda_f(n)e(un/N)}{n^w}.$$

Note that you may assume $(u,N)=1$. We also define the completed L-function as

$$\Lambda(w,f,u/N)=\pi^{-w}L(w,f,u/N)\Gamma(w+\tfrac{k-1}{2}). \tag{16}$$

Theorem 3.1. *Consider the integral*

$$I(s,w;f;\chi):=\iint_{\Gamma_0(N)\backslash\mathbb{H}}f(z)y^{\frac{k}{2}}\overline{E^{(N)}(z,\tfrac{\bar w}{2},\chi,k)}E^{(N)}(z,s+\tfrac{w}{2}-\tfrac12,\bar\chi)\frac{\mathrm{d}x\mathrm{d}y}{y^2}. \tag{17}$$

The integrand is invariant under $\Gamma_0(N)$, and can be unfolded with respect to either Eisenstein series. Considering various $\chi \mod N$ we deduce the functional equation

$$\Lambda(s,f,\tfrac{u}{N})=i^k2(N/2)^{1-2s}\Lambda(1-s,f,-\tfrac{v}{N}),$$

where $(u,N)=1, uv\equiv 1 \mod N$ and $\Lambda(w,f,\frac{u}{N})$ is the completed additively twisted L-function of f, as in (16).

Proof. On the one hand, unfolding the second Eisenstein series yields

$$\begin{aligned}I(s,w;f;\chi)=&i^{-k}\sum_{n=1}^{\infty}\sum_{c=1}^{\infty}\frac{\lambda_f(n)n^{\frac{k-1}{2}}g(\bar\chi,Nc,n)}{(Nc)^w}\frac{n^{\frac{w}{2}-1}\pi^{\frac{w}{2}}}{\Gamma(\frac{w}{2}+\frac{k}{2})}\int_0^\infty e^{-2\pi ny}W_{\frac{k}{2},\frac12-\frac{w}{2}}(4\pi ny)y^{s+\frac{w}{2}+\frac{k-1}{2}-1}\frac{\mathrm{d}y}{y}\\=&i^{-k}\sum_{n=1}^{\infty}\sum_{c=1}^{\infty}\frac{\lambda_f(n)g(\bar\chi,Nc,n)}{n^s(Nc)^w}\frac{\pi^{\frac{w}{2}}\Gamma(s+\frac{k-1}{2})\Gamma(s-w+\frac{k-1}{2}-1)}{(2\pi)^{s+\frac{w}{2}+\frac{k-1}{2}-1}\Gamma(\frac{w}{2}+\frac{k}{2})\Gamma(s+\frac{w}{2}-\frac12)2^{s+\frac{w}{2}+\frac{k-1}{2}-1}}.\end{aligned}$$

On the other hand, unfolding the first Eisenstein series yields

$$\begin{aligned}I(s,w;f;\chi)=&\sum_{n=1}^{\infty}\sum_{c=1}^{\infty}\frac{\lambda_f(n)n^{\frac{k-1}{2}}g(\chi,cN,-n)}{(cN)^{w+2s-1}}\frac{n^{s+\frac{w}{2}-1}\pi^{s+\frac{w}{2}-\frac12}}{\Gamma(s+\frac{w}{2}-1)}\int_0^\infty e^{-2\pi ny}K_{s+\frac{w}{2}-1}(2\pi ny)y^{\frac{w}{2}+\frac{k-1}{2}}\frac{\mathrm{d}y}{y}\\=&\sum_{n=1}^{\infty}\sum_{c=1}^{\infty}\frac{\lambda_f(n)g(\chi,cN,-n)}{n^{1-s}(cN)^{w+2s-1}}\frac{\pi^{s+\frac{w}{2}-\frac12}}{(2\pi)^{\frac{w}{2}+\frac{k-1}{2}}\Gamma(s+\frac{w}{2}-1)}\frac{\sqrt{\pi}\Gamma(s+w+\frac{k-1}{2}-1)\Gamma(1-s+\frac{k-1}{2})}{2^{\frac{w}{2}+\frac{k-1}{2}}\Gamma(\frac{w}{2}+\frac{k}{2})}.\end{aligned}$$

For convenience, denote

$$R(s,w)=\frac{1}{N^w}\frac{\Gamma(s+w+\frac{k-1}{2}-1)}{\Gamma(s+\frac{w}{2}-\frac12)\Gamma(\frac{w}{2}+\frac{k}{2})}\frac{\pi}{(4\pi)^{\frac{w}{2}+\frac{k-1}{2}}}.$$

Open the Gauss sum in either computation, and then one of the identities reads

$$I(s,w;f;\chi)=i^{-k}2^{1-2s}2\sum_{c=1}^{\infty}\frac{1}{c^w}\sum_{\substack{u \mod cN\\(u,cN)=1}}\overline{\chi(u)}\Lambda(s,f,u/cN)R(s,w)$$

for $\Re(w) > 1$, and the other identity reads,

$$I(s, w; f; \chi) = (cN)^{1-2s} \sum_{c=1}^{\infty} \frac{1}{c^w} \sum_{\substack{u \mod cN \\ (u,cN)=1}} \chi(u)\Lambda(1-s, f, -u/cN)R(s, w).$$

By the uniqueness of Dirichlet series, we deduce that all coefficients, and in particular the coefficient corresponding to $c = 1$, are identical. Cancelling the R factor we get,

$$\sum_{u \mod N} \overline{\chi(u)}\Lambda(s, f, \frac{u}{N}) = i^k 2(N/2)^{1-2s} \sum_{u \mod N} \chi(u)\Lambda(1-s, f, -\tfrac{u}{N}).$$

Notice that this is true for any character $\chi \mod N$, and hence using the orthogonality relations for Dirichlet characters we may extract a single term. Multiply both sides by $\frac{1}{\phi(N)}\chi(v)$ and sum over $\chi \mod N$. On the left hand side we are left with only the $u = v$ term, whereas on the right hand side we are left with the u satisfying $-uv \equiv 1 \mod N$.

$$\Lambda(s, f, \tfrac{v}{N}) = i^k 2(N/2)^{1-2s}\Lambda(1-s, f, -\tfrac{\bar{v}}{N}).$$

The factors of 2 in the above functional equation clear themselves out if we apply Legendre duplication formula to the Gamma functions on either side. □

3.2 Double Dirichlet series for GL(2)

In this subsection we give another proof of the Voronoi formula on GL(2) for Hecke-Maass cusp forms for $\mathrm{SL}(2, \mathbb{Z})$ that does not include unfolding. It illustrates the main strategy of our proof for $\mathrm{GL}(N)$.

Theorem 3.2. *Let f be a Hecke-Maass cusp form for $SL(2, \mathbb{Z})$ with Hecke eigenvalue $A(n)$. Let $c > 1$ be an integer and let d be any integer with $(d, c) = 1$. Denote by $\bar{d}$ the multiplicative inverse of d modulo c. Define a Dirichlet series*

$$L\left(s, f, \tfrac{d}{c}\right) := \sum_{n=1}^{\infty} \frac{A(n)}{n^s} e\left(\frac{dn}{c}\right),$$

which is absolutely convergence for $\Re(s) > 1$, has analytic continuation to the whole complex plane and satisfies the functional equation

$$L\left(s, f, \tfrac{\bar{d}}{c}\right) = \frac{G_+(s) + G_-(s)}{2} L\left(1-s, f, \tfrac{d}{c}\right) + \frac{G_+(s) - G_-(s)}{2} L\left(1-s, f, \tfrac{-d}{c}\right), \tag{18}$$

where $G_{\pm}(s)$ is defined in (6).

Take any character χ modulo c and assume that it has been induced from the primitive character χ^* modulo c^*. We have $c^*|c$. Multiply both sides of (18) by $\chi(d)$ and sum over d modulo c. The formula (18) ends up being equivalent to,

$$\sum_n \frac{A(n)g(\overline{\chi^*}, c, n)}{n^s} = G(s)c^{1-2s} \sum_m \frac{A(m)g(\chi^*, c, m)}{m^{1-s}}, \tag{19}$$

where an equality of the analtyic continuations of either side is implied, and where $G(s)$ equals $G_+(s)$ or $G_-(s)$ depending on whether χ^* is even or odd. Also $g(\chi^*, c, m)$ denotes the Gauss sum of χ as in Definition 2.1.

Proof of (19) *and Theorem 3.2.* We start with the expression

$$\frac{L(s + 2w, f)L(s, f \times \chi^*)}{L(2w + 2s, \chi^*)},$$

and expand it using their Dirichlet series expression and simplify the resulting expression using Hecke relations,

$$\begin{aligned} \frac{L(s + 2w, f)L(s, f \times \chi^*)}{L(2w + 2s, \chi^*)} &= \sum_{n=1} \frac{A(n)\sigma_{2w}(n, \chi^*)}{n^{s+2w}} \\ &= L(1 + 2w, \overline{\chi^*})\tau(\overline{\chi^*})^{-1} \sum_{n=1}^{\infty} \sum_{\substack{c=1 \\ c^*|c}}^{\infty} \frac{A(n)}{n^s} \frac{g(\overline{\chi^*}, c, n)}{(c/c^*)^{1+2w}}. \end{aligned}$$

Here $\sigma_s(m,\chi^*)$ is defined in Definition 2.1, and in the second line we used the expansion in terms of Gauss sums as in Lemma 2.4.

On the other hand, using the same methods

$$\frac{L(s+2w,f)L(1-s,f\times\overline{\chi^*})}{L(2w+2s,\chi^*)}=L(1+2w,\overline{\chi^*})\tau(\chi^*)^{-1}\sum_{n=1}^{\infty}\sum_{\substack{c=1\\c^*|c}}^{\infty}\frac{A(n)}{n^{1-s}}\frac{g(\chi^*,c,n)}{(c/c^*)^{2w+2s}}.$$

These two are related as the L-function of $f\times\chi^*$ satisfies the functional equation

$$L(s,f\times\chi^*)=\tau(\chi^*)^2{c^*}^{-2s}G(s)L(1-s,f\times\overline{\chi^*}),\tag{20}$$

where $G(s)$ equals $G_+(s)$ or $G_-(s)$ depending on whether $\chi^*(-1)$ is 1 or -1.

By the functional equation (20), we have

$$\sum_{\substack{c=1\\c^*|c}}^{\infty}\frac{1}{c^{2w}}\sum_{n=1}^{\infty}\frac{A(n)g(\overline{\chi^*},c,n)}{n^sc}=G(s)\sum_{\substack{c=1\\c^*|c}}^{\infty}\frac{1}{c^{2w}}\sum_{n=1}^{\infty}\frac{A(n)g(\chi^*,c,n)}{n^{1-s}c^{2s}}$$

Applying uniqueness of Dirichlet coefficients in the variable w, we get (19) as well as Theorem 3.2. $\square$

Notice that in Section 3.1 we used the Rankin-Selberg integral to study the L-function arising from $F\times E(*,\chi^*)$. The transformation $s\to 1-s$ was not apparent from looking at the integral $I(s,w)$, but came from unfolding different Eisenstein series. Instead in Section 3.2 we start directly with the L-function. There are advantages and disadvantages of either approach, but in higher rank groups, the advantages of the latter are more obvious.

Firstly note that in the integral approach, there is no mention of Hecke operators or Hecke relations. Therefore f is not necessarily a Hecke eigenform, this only becomes a real advantage when working metaplectically. We only use the automorphy of f in the unfolding process, and the functional equation arises "on its own". In the L-function approach we do not use automorphy and are able to generalize our result to all functions that satisfy the standard twisted functional equation (15).

4 Voronoi Formula on GL(3)

Theorem 4.1 (Voronoi formula on GL(3) of Miller-Schmid [MS06])**.** *Let $c>1$ be an integer and let d be any integer with $(d,c)=1$. Denote by $\bar{d}$ the multiplicative inverse of d modulo c. Let F be a Hecke-Maass cusp form for $SL(3,\mathbb{Z})$ with Fourier coefficient $A(*,*)$. Let $G_\pm(s)$ be defined as in* (6) *with $N=3$. Notate*

$$L\left(s,F,m,\frac{d}{c}\right)=\sum_{n=1}^{\infty}\frac{A(m,n)}{n^s}e\left(\frac{n\bar{d}}{c}\right).$$

This function $L(s,F,m,d/c)$ has an analytic continuation to the entire s-plane and satisfies the identity

$$\begin{aligned}L\left(s,F,m,\frac{d}{c}\right)=&\frac{G_+(s)+G_-(s)}{2}\sum_{n_2}\sum_{n_1|mc}\frac{cA(n_2,n_1)}{n_1n_2(n_2n_1^2/c^3m)^{-s}}S(md,n_2;cm/n_1)\\&+\frac{G_+(s)-G_-(s)}{2}\sum_{n_2}\sum_{n_1|mc}\frac{cA(n_2,n_1)}{n_1n_2(n_2n_1^2/c^3m)^{-s}}S(md,-n_2;cm/n_1),\end{aligned}\tag{21}$$

in the region of absolute convergence of the right hand side.

This theorem is equivalent with the following one.

Theorem 4.2 (Average Voronoi formula with character on GL(3))**.** *Let χ be any Dirichlet character mod c induced from a primitive Dirichlet character χ^* mod c^*. Define*

$$H(m,c,\chi^*,s)=\sum_{n=1}^{\infty}\frac{A(m,n)g(\overline{\chi^*},c,n)}{n^s(c/c^*)^{1-2s}}.$$

and

$$G(m,c,\chi^*,s)=\frac{G(s)}{c^{*3s-1}}\sum_{n_2}\sum_{n_1c^*|mc}\frac{A(n_2,n_1)g(\chi^*,c,n_1)g(\chi^*,mc/n_1,n_2)}{n_2^{1-s}n_1^{1-2s}m^s(c/c^*)^s},$$

where the function $G(s)$ is given by $G_+(s)$ or $G_-(s)$ depending on whether χ^* is even or odd. The average Voronoi formula is

$$H(m,c,\chi^*,s)=G(m,c,\chi^*,s), \tag{22}$$

with the understanding that the left side has analytic continuation to the whole complex plane.

Proof of Theorem 4.1 and 4.2. We first prove the identity in Theorem 4.8. Then in order to have the equality for a single term $H(m,c,\chi^*,s)=G(m,c,\chi^*,s)$, we run an induction and prove Theorem 4.2, the same induction performed in the proof of Theorem 1.2 in Subsection 5.2. Theorem 4.1 is equivalent to Theorem 4.2 (by Theorem 4.3) and hence we have proven that as well. □

4.1 Equivalence of Theorems 4.1 and 4.2.

Theorem 4.3. *Theorem 4.1 is equivalent to Theorem 4.2.*

Lemma 4.4. *Let χ be a Dirichlet character mod c induced from a primitive Dirichlet character χ^* mod c^*, and let m,d,n_1,n_2 be integers such that $n_1|cm$. Define*

$$S=\sum_{\substack{d \mod c\\(d,c)=1}}\chi(d)S(md,n_2;mc/n_1).$$

We have

$$S=\begin{cases}g(\chi^*,c,n_1)g(\chi^*,mc/n_1,n_2), & \text{if } n_1c^*|mc,\\ \\ 0, & \text{otherwise.}\end{cases}$$

Proof. By Lemma 2.2, we have

$$\begin{aligned}S&=\sum_{\substack{d \mod c\\(d,c)=1}}\sum_{\substack{x \mod mc/n_1\\(x,mc/n_1)=1}}e\left(\frac{mdx}{mc/n_1}\right)e\left(\frac{n_2\bar{x}}{mc/n_1}\right)\chi^*(d)\\&=\sum_{\substack{x \mod mc/n_1\\(x,mc/n_1)=1}}e\left(\frac{n_2\bar{x}}{mc/n_1}\right)\sum_{\substack{d \mod c\\(d,c)=1}}\chi^*(d)e\left(\frac{mdx}{mc/n_1}\right)\\&=\sum_{\substack{x \mod mc/n_1\\(x,mc/n_1)=1}}e\left(\frac{n_2\bar{x}}{mc/n_1}\right)g(\chi^*,c,n_1x).\end{aligned} \tag{23}$$

As $(x,mc/n_1)=1$, we have $(n_1x,mc)=n_1$. Therefore, $(c,n_1x)=(c,n_1x,mc)=(c,(n_1x,mc))=(c,n_1)$. Hence 2.3 implies the identity $g(\chi^*,c,n_1x)=\overline{\chi^*}(x)g(\chi^*,c,n_1)$.

We are given $n_1|mc$, and thus $n_1/(c,n_1)|mc/(c,n_1)$, but since $n_1/(c,n_1)$ and $c/(c,n_1)$ are coprime, $n_1/(c,n_1)|m$. By Lemma 2.3, for $g(\chi^*,c,n_1)$ to be nonzero, we must have $c^*|c/(c,n_1)$. Consequently, we have

$$n_1c^*\left|n_1\frac{c}{(c,n_1)}\right|mc.$$

In such a case multiplicative inverses modulo mc/n_1 and modulo c^* are compatible as long as they both make sense and we compute

$$\begin{aligned}S&=\sum_{\substack{x \mod mc/n_1\\(x,mc/n_1)=1}}e\left(\frac{n_2\bar{x}}{mc/n_1}\right)\overline{\chi^*}(x)g(\chi^*,c,n_1)\\&=g(\chi^*,mc/n_1,n_2)g(\chi^*,c,n_1).\end{aligned}$$

□

Proof of Theorem 4.3. Let us see Theorem 4.1 implies Theorem 4.2 firstly. Multiply both sides of (21) by $\chi(d)$ and sum d over reduced residue classes modulo c. Then using Lemma 4.4 we obtain,

$$\begin{aligned}
&\sum_{\substack{(d,c)=1 \\ d \mod c}} \sum_{n} \frac{A(m,n)}{n^s} e\left(\frac{n\bar{d}}{c}\right)\chi(d) \\
&\quad = \frac{G_+(s)+G_-(s)}{2} \sum_{n_2}\sum_{n_1c^*|mc} \frac{cA(n_2,n_1)}{n_1n_2(n_2n_1^2/c^3m)^{-s}} \tau(\chi^*) g(\chi^*,c,n_1) g(\chi^*, mc/n_1, n_2) \\
&\quad\quad + \frac{G_+(s)-G_-(s)}{2} \sum_{n_2}\sum_{n_1c^*|mc} \frac{cA(n_2,n_1)}{n_1n_2(n_2n_1^2/c^3m)^{-s}} \tau(\chi^*) g(\chi^*,c,n_1) g(\chi^*, mc/n_1, -n_2).
\end{aligned}$$

If χ^* is an even character, i.e. if $\chi^*(-1)=1$, we have $g(\chi^*, mc/n_1, -n_2) = g(\chi^*, mc/n_1, n_2)$. Otherwise, $g(\chi^*, mc/n_1, -n_2) = -g(\chi^*, mc/n_1, n_2)$. This cancels either G_+ or G_- and we obtain Theorem 4.2.

Conversely, and more importantly Theorem 4.1 implies Theorem 4.2 by the orthogonality relations for Dirichlet characters. □

4.2 Instructional Example: Special Case of $c = 1$

The Fourier coefficients Eisenstein series $E(z,s)$ can be given as a Dirichlet series with the introduction of a factor of $\zeta(2s)$. Equivalently we use an identity of Ramanujan

$$\frac{\sigma_{s-1}(n)}{\zeta(s)} = n^{s-1}\sum_{\ell=1}^{\infty}\frac{c_\ell(n)}{\ell^s}, \tag{24}$$

where $c_\ell(n) = g(\mathbf{1},\ell,n)$ is the Ramanujan sum, in order to obtain the divisor functions. In this subsection we are going through some computations involved in the proof of the Voronoi formula without dealing with characters.

In Theorem 4.1, if we take $c = 1$, the Kloosterman sum $S(md, n_2; m/n_1)$ collapses into a Ramanujan sum $c_{m/n_1}(n_2)$ because

$$S(md,n_2;m/n_1) = \sum_{\substack{u \mod m/n_1 \\ (u,m/n_1)=1}} e\left(\frac{md\bar{u}}{m/n_1}\right) e\left(\frac{un_2}{m/n_1}\right) = \sum_{\substack{u \mod m/n_1 \\ (u,m/n_1)=1}} e\left(\frac{un_2}{m/n_1}\right) = c_{m/n_1}(n_2).$$

Theorem 4.5 (Bump's double Dirichlet series)**.** *The Bump's double Dirichlet series has the factorization*

$$\sum_{m_1}\sum_{m_2}\frac{A(m_1,m_2)}{m_1^t m_2^s} = \frac{L(t,\tilde{F})L(s,F)}{\zeta(t+s)}.$$

Proof. See Proposition 6.6.3 of [Gol06] or Chapter X of [Bum84]. □

Lemma 4.6. *The products of two shifted L-function of F gives rise to divisor functions in the form of the following identity between the Dirichlet series*

$$L(s+a,F)L(s-a,F) = \sum_{m,n=1}^{\infty}\frac{A(m,n)}{m^{2s}n^s}\frac{\sigma_{2a}(n)}{n^a}.$$

Proof. The proof is a short computation. Using the Hecke relations,

$$\begin{aligned}
L(s+a,F)L(s-a,F) &= \sum_{m,n=1}^{\infty}\sum_{d|(m,n)}\frac{A(d,mn/d^2)}{n^{s+a}m^{s-a}} \\
&= \sum_{d=1}^{\infty}\sum_{h=1}^{\infty}\sum_{h_1h_2=h}\frac{A(d,h)}{d^{2s}h^s}\frac{h_1^a}{h_2^a}.
\end{aligned}$$

□

Theorem 4.7 (Special case of Voronoi formula $c = 1$)**.** *Let $A(*,*)$ be the Fourier-Whittaker coefficient of a Hecke-Maass cusp form for $SL(3,\mathbb{Z})$. For a fixed integer $m > 0$, we have*

$$\sum_{n=1}^{\infty} \frac{A(m,n)}{n^s} = G_+(s) \sum_{n_2=1}^{\infty} \sum_{n_1|m} \frac{A(n_2,n_1)c_{m/n_1}(n_2)}{n_1^{1-2s} n_2^{1-s} m^s},$$

with the understanding that the left side is absolutely convergent for $\Re(s) \gg 1$ and has analytic continuation to the whole complex plane.

Proof. After applying the functional equation 15 for the variable s we have

$$\begin{aligned}
\sum_{m,n} \frac{A(m,n)}{m^t n^s} &= \frac{L(t,\tilde{F})L(s,F)}{\zeta(t+s)} \\
&= G_+(s)L(t,\tilde{F})L(1-s,\tilde{F})/\zeta(t+s) \\
&= G_+(s)L(s+\delta,\tilde{F})L(s-\delta,\tilde{F})/\zeta(1+2\delta),
\end{aligned}$$

where $s = (1-s+t)/2$ and $\delta = (t-1+s)/2$. Applying (24), we have by Lemma 4.6

$$\begin{aligned}
\sum_{m,n} \frac{A(m,n)}{m^t n^s} &= G_+(s) \sum_{n_1,n_2} \frac{A(n_2,n_1)}{n_1^{2s} n_2^{s}} \frac{\sigma_{2\delta}(n_2)}{n_2^{\delta}} \frac{1}{\zeta(1+2\delta)} \\
&= G_+(s) \sum_{n_1,n_2} \sum_{c} \frac{A(n_2,n_1)}{n_1^{1-s+t} n_2^{1-s}} \frac{c_c(n_2)}{c^{t+s}}.
\end{aligned}$$

Hence, we get the average Voronoi formula after applying uniqueness of Dirichlet series coefficients with variable t. □

4.3 Main Computation on GL(3)

We first obtain the desired equality for a convolution sum, and then we will be able to obtain the equality for a single term.

Theorem 4.8. *Let χ be a Dirichlet character mod c induced from a primitive Dirichlet character χ^* mod c^*. For any fixed positive integer n, and any positive integer q we have*

$$\sum_{e|q} \frac{\chi^*(e)}{e^s} \sum_{\ell d=n} \chi^*(d) H(\tfrac{q}{e}d, \ell c^*, \chi^*) = \sum_{e|q} \frac{\chi^*(e)}{e^s} \sum_{\ell d=n} \chi^*(d) G(\tfrac{q}{e}d, \ell c^*, \chi^*),$$

with the understanding that the left side is absolutely convergent for $\Re(s) \gg 1$ and has analytic continuation to the whole complex plane.

Proof. Let q be any positive integer. We define a Dirichlet series,

$$L_q(s,F) = \sum_{n=1}^{\infty} \frac{A(q,n)}{n^s}.$$

We begin our considerations with the function,

$$Z(s,w) := \frac{L_q(2w-s,F)L(s,F\times\chi^*)}{L(2w-2s+1,\overline{\chi^*})},$$

where χ^* is a primitive character with conductor c^*. The function $Z(s,w)$ is meromorphic of both variables with the only poles coming from the zeros of the Dirichlet L-function in the denominator.

Now let us assume that $\Re(s) > 1$ and $\Re(w-s) > 0$, so that we can expand both L-functions in Dirichlet series and apply Hecke relations to the product. We have

$$\begin{aligned}
Z(s,w) &= L(2w-2s+1,\overline{\chi^*})^{-1} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{A(q,n)A(1,m)\chi^*(m)}{n^{2w-s}m^s} \\
&= L(2w-2s+1,\overline{\chi^*})^{-1} \sum_{n,m=1}^{\infty} \sum_{\substack{d_0d_1d_2=m \\ d_1|n, d_2|q}} \frac{A\left(\frac{qd_1}{d_2}, \frac{nd_0}{d_1}\right)\chi^*(m)}{n^{2w-s}m^s},
\end{aligned}$$

in which we use (3). Continuing the computation, we have

$$\begin{aligned}Z(s,w) &= L(2w-2s+1,\overline{\chi^*})^{-1}\sum_{n,m=1}^{\infty}\sum_{\substack{d_0d_1d_2=m\\ d_1|n,d_2|q}}\frac{A\left(\frac{qd_1}{d_2},\frac{n}{d_1}d_0\right)\chi^*(d_0d_1d_2)}{d_1^{2w}d_2^s(nd_0/d_1)^{2w-s}d_0^{2s-2w}}\\ &= L(2w-2s+1,\overline{\chi^*})^{-1}\sum_{n,m=1}^{\infty}\sum_{\substack{d_0d_1d_2=m\\ d_1|n,d_2|q}}\frac{\chi^*(d_1d_2)A\left(\frac{qd_1}{d_2},\frac{n}{d_1}d_0\right)\chi^*(d_0)}{d_1^{2w}d_2^s(nd_0/d_1)^{2w-s}d_0^{2s-2w}}.\end{aligned}$$

We make a change of variables $d_0n\mapsto n$,

$$\begin{aligned}Z(s,w) &= L(2w-2s+1,\overline{\chi^*})^{-1}\sum_{d_1,n=1}^{\infty}\sum_{\substack{d_0|n\\ d_2|q}}\frac{\chi^*(d_1d_2)A\left(\frac{qd_1}{d_2},n\right)\chi^*(d_0)}{d_1^{2w}d_2^s n^{2w-s}d_0^{2s-2w}}\\ &= L(2w-2s+1,\overline{\chi^*})^{-1}\sum_{d_1,n=1}^{\infty}\sum_{d_2|q}\frac{\chi^*(d_1d_2)A\left(\frac{qd_1}{d_2},n\right)}{d_1^{2w}d_2^s n^s}\frac{\sigma_{2w-2s}(n,\chi^*)}{n^{2w-2s}}.\end{aligned}$$

We expand this character-twisted divisor function into a Dirichlet series of Gauss sums with the help of Lemma (2.4),

$$Z(s,w)=\tau(\overline{\chi^*})^{-1}\sum_{d_2|q}\frac{\chi^*(d_2)}{d_2^s}\sum_{d=1}^{\infty}\chi^*(d)\sum_{h=1}^{\infty}\frac{A\left(\frac{q}{d_2}d,h\right)}{d^{2w}h^s}\sum_{\ell=1}^{\infty}\frac{g(\overline{\chi^*},\ell c^*,h)}{\ell^{2w-2s+1}}.$$

Finally we plug in the definiton of H and combine $c\ell=n$.

$$Z(s,w)=\tau(\overline{\chi^*})^{-1}\sum_{n=1}^{\infty}\frac{1}{n^{2w}}\sum_{d_2|q}\frac{\chi^*(d_2)}{d_2^s}\sum_{\ell d=n}\chi^*(d)H(\tfrac{q}{d_2}d,\ell c^*,\chi^*,s). \tag{25}$$

Although initially we assumed that $\Re(s)>1$ in order to do these computations, the s-function

$$\sum_{d_2|q}\frac{\chi^*(d_2)}{d_2^s}\sum_{\ell d=n}\chi^*(d)H(\tfrac{q}{d_2}d,\ell c^*,\chi^*,s)$$

has analytic continuation to the whole complex plane.

Now we make use of the functional equation of $L(s,F\times\chi^*)$ in the numerator of $Z(s,w)$,

$$L(s,F\times\chi^*)=\tau(\chi^*)^3c^{*-3s}G(s)L(1-s,\widetilde{F}\times\overline{\chi^*}),$$

where $G(s)=G_\pm(s)$ depending on whether χ^* is an even or an odd Dirichlet character. This time assuming $\Re(2w-s)>1$, $\Re(s)<0$, we go through the same motions,

$$\begin{aligned}Z(s,w) =&\frac{G(s)\tau(\chi^*)^3c^{*-3s}}{L(2w-2s+1,\overline{\chi^*})}L_q(2w-s,F)L(1-s,\tilde{F}\times\overline{\chi^*})\\ =&\frac{G(s)\tau(\chi^*)^3c^{*-3s}}{L(2w-2s+1,\overline{\chi^*})}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{A(q,n)A(m,1)\overline{\chi^*(m)}}{n^{2w-s}m^{1-s}}\\ =&\frac{G(s)\tau(\chi^*)^3c^{*-3s}}{L(2w-2s+1,\overline{\chi^*})}\sum_{n,m=1}^{\infty}\sum_{\substack{d_0d_1d_2=m\\ d_1|q,d_2|n}}\frac{A\left(\frac{q}{d_1}d_0,\frac{nd_1}{d_2}\right)\overline{\chi^*(m)}}{n^{2w-s}m^{1-s}}\\ =&\frac{G(s)\tau(\chi^*)^3c^{*-3s}}{L(2w-2s+1,\overline{\chi^*})}\sum_{d_1|q}\frac{\overline{\chi^*(d_1)}}{d_1^{1-s}}\sum_{d_2=1}^{\infty}\frac{\overline{\chi^*(d_2)}}{d_2^{2w-2s+1}}\sum_{m,n=1}^{\infty}\frac{A\left(\frac{q}{d_1}m,nd_1\right)\overline{\chi^*(m)}}{n^{2w-s}m^{1-s}}.\end{aligned}$$

Here note that the sum over d_2 gives the Dirichlet L-function of the character $\overline{\chi^*}$ which cancels with the denominator. Changing the name of some variables,

$$Z(s,w)=\frac{G(s)\tau(\chi^*)^2}{\tau(\overline{\chi^*})c^{*3s-1}}\sum_{n=1}^{\infty}\frac{1}{n^{2w}}\sum_{e|q}\frac{\overline{\chi^*(e)}}{e^{1-s}}\sum_{m=1}^{\infty}\frac{A\left(\frac{q}{e}m,ne\right)\overline{\chi^*(m)}n^s}{m^{1-s}}. \tag{26}$$

At this point, it is easier to start from the desired formulation, and work backwards to this expression. We would like to have (26) equal to,

$$\begin{aligned}
&\tau(\overline{\chi^*})^{-1}\sum_{n=1}^{\infty}\frac{1}{n^{2w}}\sum_{e|q}\frac{\chi^*(e)}{e^s}\sum_{\ell d=n}\chi^*(d)G(\tfrac{q}{e}d,\ell c^*,\chi^*,s)\\
&\quad=\frac{G(s)}{\tau(\overline{\chi^*})c^{*3s-1}}\sum_{n=1}^{\infty}\frac{1}{n^{2w}}\sum_{e|q}\frac{\chi^*(e)}{e^s}\sum_{\ell d=n}\chi^*(d)\\
&\quad\quad\times\sum_{n_2=1}^{\infty}\sum_{n_1|\frac{qn}{e}}\frac{A(n_2,n_1)g(\chi^*,c,n_1)g(\chi^*,qnc^*/en_1,n_2)}{n_2^{1-s}n_1^{1-2s}(qn/e)^s}.
\end{aligned}$$

For this equality to occur, the Dirichlet series in the n variable must be equal to each other. Therefore we would need to prove that their coefficients are equal one by one. Define

$$L(n)=\tau(\chi^*)^2\sum_{e|q}\frac{\overline{\chi^*(e)}}{e^{1-s}}\sum_{m=1}^{\infty}\frac{A\left(\frac{q}{e}m,ne/c^*\right)\overline{\chi^*(m)}n^s}{m^{1-s}},$$

and

$$R(n)=\sum_{e|q}\frac{\chi^*(e)}{e^s}\sum_{\ell d=n}\chi^*(d)\sum_{n_2=1}^{\infty}\sum_{n_1|\frac{qn}{e}}\frac{A(n_2,n_1)g(\chi^*,\ell c^*,n_1)g(\chi^*,qnc^*/en_1,n_2)}{n_2^{1-s}n_1^{1-2s}(qn/e)^s}.$$

Our goal is to have $L(n)=R(n)$.

Now use Lemma 2.5 to the sum over the set of ℓ and d's with $\ell d=n$. Let $\delta_{n|n_1}$ be the Kronecker symbol for whether n divides n_1. Then, we have

$$\begin{aligned}
R(n)=&\sum_{e|q}\frac{\chi^*(e)}{e^s}\sum_{n_2=1}^{\infty}\sum_{n_1|\frac{qn}{e}}\frac{A(n_2,n_1)g(\chi^*,qnc^*/en_1,n_2)}{n_2^{1-s}n_1^{1-2s}(qn/e)^s}\tau(\chi^*)\delta_{n|n_1}\overline{\chi^*(n_1/n)}n\\
=&\sum_{e|q}\chi^*(e)\sum_{f|\frac{q}{e}}\sum_{n_2=1}^{\infty}\frac{A(n_2,fn)\tau(\chi^*)\overline{\chi^*(f)}g(\chi^*,\frac{q/e}{f}c^*,n_2)}{n_2^{1-s}n^{-s}q^sf^{1-2s}},
\end{aligned}$$

where we named $f=n_1/n$, since n_1 runs over integers dividing qn/e and since $n|n_1$, the quantity f runs over integers dividing q/e. We change the order of summation in f and e, and then apply Lemma 2.5 once again,

$$\begin{aligned}
R(n)=&\sum_{f|q}\sum_{e|\frac{q}{f}}\chi^*(e)\overline{\chi^*(f)}\sum_{n_2=1}^{\infty}\frac{A\left(n_2,fn\right)\tau(\chi^*)g(\chi^*,\frac{q/f}{e}c^*,n_2)}{n_2^{1-s}n^{-s}q^sf^{1-2s}}\\
=&\sum_{f|q}\overline{\chi^*(f)}\sum_{n_2=1}^{\infty}\frac{A\left(n_2,fn\right)\tau(\chi^*)^2}{n_2^{1-s}n^{-s}q^sf^{1-2s}}\delta_{\frac{q}{f}|n_2}\overline{\chi^*\left(\frac{n_2}{q/f}\right)}\frac{q}{f}.
\end{aligned}$$

We denote $m=n_2/(q/f)$ and that gives,

$$\begin{aligned}
R(n)=&\sum_{f|q}\overline{\chi^*(f)}\sum_{m=1}^{\infty}\frac{A\left(m\frac{q}{f},nf\right)\tau(\chi^*)^2\overline{\chi^*(m)}n^s}{m^{1-s}\left(\frac{q}{f}\right)^{1-s}q^sf^{1-2s}}\frac{q}{f}\\
=&\sum_{f|q}\frac{\overline{\chi^*(f)}}{f^{1-s}}\sum_{m=1}^{\infty}\frac{A\left(m\frac{q}{f},nf\right)n^s\tau(\chi^*)^2\overline{\chi^*(m)}}{m^{1-s}}.
\end{aligned}$$

This gives us $R(n) = L(n)$. Hence we have obtained the desired equality

$$Z(s,w) = \tau(\overline{\chi^*})^{-1} \sum_{n=1}^{\infty} \frac{1}{n^{2w}} \sum_{d_2|q} \frac{\chi^*(d_2)}{d_2^s} \sum_{\ell d=n} \chi^*(d) H(\tfrac{q}{d_2} d, \ell c^*, \chi^*, s).$$

Comparing this with (25), by uniqueness of Dirichlet coefficients, we complete the proof. □

5 The Voronoi Formula

5.1 Double Dirichlet Series

We first prove the following equality of convolution sums, from which we prove the equality (11) in Section 5.2.

Theorem 5.1. *Let $N \geq 3$. For positive integers $q_1, \ldots, q_{N-2}, n$, we have*

$$\sum_{d_1|q_1,\ldots,d_{N-2}|q_{N-2}} \frac{\chi^*(d_1 \cdots d_{N-2})}{(d_1 \cdots d_{N-2})^s} \sum_{d\ell=n} \chi^*(d) H(\mathbf{q}', \ell c^*, \chi^*, s)$$
$$= \sum_{d_1|q_1,\ldots,d_{N-2}|q_{N-2}} \frac{\chi^*(d_1 \cdots d_{N-2})}{(d_1 \cdots d_{N-2})^s} \sum_{d\ell=n} \chi^*(d) G(\mathbf{q}', \ell c^*, \chi^*, s), \quad (27)$$

where we denote for abbreviation

$$\mathbf{q}' = (\tfrac{q_1 d}{d_1}, \tfrac{q_2 d_1}{d_2}, \ldots, \tfrac{q_{N-2} d_{N-3}}{d_{N-2}}). \quad (28)$$

Both sides of the equality have analytic continuation to the whole complex plane.

Proof. Let $Z(s,w)$ be defined as in (12). Writing $L_{\mathbf{q}}(2w-s, F)$ and $L(2w-2s+1, \overline{\chi^*})^{-1}$ as Dirichlet series, we derive

$$Z(s,w) = L(s, F \times \chi^*) \sum_{n=1}^{\infty} \frac{\sum_{d|n} A(q_{N-2}, \ldots, q_1, d) d^s \overline{\chi^*}(n/d) \mu(n/d) (n/d)^{2s-1}}{n^{2w}}.$$

For s with $\Re(s)$ bounded, for $\Re(w) \gg 1$, we have

$$Z(s,w) = \sum_{n=1}^{\infty} \frac{a_n(s)}{n^{2w}}.$$

where

$$a_n(s) = L(s, F \times \chi^*) \sum_{d|n} A(q_{N-2}, \ldots, q_1, d) d^s \overline{\chi^*}(n/d) \mu(n/d) (n/d)^{2s-1}.$$

Here $a_n(s)$ has analytic continuation to the whole complex plane, because $L(s, F \times \chi^*)$ is entire. Computation below shows that $a_n(s)$ equals either side of (27), up to scaling by a constant $\tau(\overline{\chi^*})$.

For $\Re(s) \gg 1, \Re(w-s) \gg 1$, we expand the two L-functions in the numerator of $Z(s,w)$ as Dirichlet series, obtaining

$$Z(s,w) = \frac{1}{L(2w-2s+1, \overline{\chi^*})} \sum_{n,m=1}^{\infty} \frac{A(q_{N-2}, \ldots, q_1, n) A(1, \ldots, 1, m) \chi^*(m)}{n^{2w-s} m^s}$$
$$= \frac{1}{L(2w-2s+1, \overline{\chi^*})} \sum_{n,m=1}^{\infty} \frac{\chi^*(m)}{n^{2w-s} m^s} \sum_{\substack{d_0 d_1 \cdots d_{N-1} = m \\ d_0|n, d_1|q_1, \ldots, d_{N-2}|q_{N-2}}} A\left(\tfrac{q_{N-2} d_{N-3}}{d_{N-2}}, \ldots, \tfrac{q_1 d_0}{d_1}, \tfrac{n d_{N-1}}{d_0}\right),$$

where we have used the the Hecke relation (3). We change the variable $n/d_0 \to n$ and combine $h = nd_{N-1}$, giving

$$\begin{aligned}
Z(s,w) &= \frac{1}{L(2w-2s+1,\overline{\chi^*})} \sum_{n,d_0,d_{N-1}=1}^{\infty} \sum_{\substack{d_i|q_i\\ i=1,\dots,N-2}} \frac{\chi^*(d_0\dots d_{N-1})}{n^{2w-s}d_0^{2w-s}(d_0\dots d_{N-1})^s} \\
&\qquad \times A\left(\tfrac{q_{N-2}d_{N-3}}{d_{N-2}},\dots,\tfrac{q_1 d_0}{d_1}, nd_{N-1}\right) \\
&= \frac{1}{L(2w-2s+1,\overline{\chi^*})} \sum_{d_0,h=1}^{\infty} \sum_{\substack{d_i|q_i\\ i=1,\dots,N-2}} \frac{\chi^*(d_0\dots d_{N-2})}{d_0^{2w-s}(d_0\dots d_{N-2})^s} \\
&\qquad \times A\left(\tfrac{q_{N-2}d_{N-3}}{d_{N-2}},\dots,\tfrac{q_1 d_0}{d_1}, h\right)\frac{\sigma_{2w-2s}(h,\chi^*)}{h^{2w-s}}.
\end{aligned}$$

Applying Lemma 2.4, we get

$$Z(s,w) = \tau(\overline{\chi^*})^{-1} \sum_{d_0=1}^{\infty} \sum_{\substack{d_i|q_i\\ i=1,\dots,N-2}} \frac{\chi^*(d_0\dots d_{N-2})}{d_0^{2w}(d_1\dots d_{N-2})^s} \sum_{h=1}^{\infty} \frac{A\left(\tfrac{q_{N-2}d_{N-3}}{d_{N-2}},\dots,\tfrac{q_1 d_0}{d_1}, h\right)}{h^s} \sum_{\ell=1}^{\infty} \frac{g(\overline{\chi^*},\ell c^*,h)}{\ell^{2w-2s+1}}.$$

Therefore we reach

$$Z(s,w) = \tau(\overline{\chi^*})^{-1} \sum_{n=1}^{\infty} \frac{1}{n^{2w}} \sum_{d_1|q_1,\dots,d_{N-2}|q_{N-2}} \frac{\chi^*(d_1\cdots d_{N-2})}{(d_1\cdots d_{N-2})^s} \sum_{d\ell=n} \chi^*(d) H(\mathbf{q}',\ell c^*,\chi^*,s), \tag{29}$$

where $\mathbf{q}'$ is defined in (28).

On the other hand, let us apply the functional equation (15), giving

$$Z(s,w) = \frac{G(s)\tau(\chi^*)^N}{c^{*Ns}} \frac{L_{\mathbf{q}}(2w-s,F)L(1-s,\widetilde{F}\times\overline{\chi^*})}{L(2w-2s+1,\overline{\chi^*})}.$$

Given $-\Re(s) \gg 1$ and $\Re(2w-s) \gg 1$, we open the expression as a Dirichlet series,

$$\begin{aligned}
Z(s,w) =& \frac{G(s)\tau(\chi^*)^N c^{*-Ns}}{L(2w-2s+1,\overline{\chi^*})} \sum_{n,m=1}^{\infty} \frac{A(q_{N-2},\dots,q_1,n)A(m,1,\dots,1)\overline{\chi^*}(m)}{n^{2w-s}m^{1-s}} \\
=& \frac{G(s)\tau(\chi^*)^N c^{*-Ns}}{L(2w-2s+1,\overline{\chi^*})} \sum_{n,m=1}^{\infty} \frac{\overline{\chi^*}(m)}{n^{2w-s}m^{1-s}} \sum_{\substack{d_0d_1\cdots d_{N-1}=m\\ d_0|n,d_1|q_1,\dots,d_{N-2}|q_{N-2}}} A\left(\tfrac{q_{N-2}d_{N-1}}{d_{N-2}},\dots,\tfrac{q_1d_2}{d_1},\tfrac{nd_1}{d_0}\right) \\
=& \frac{G(s)\tau(\chi^*)^N c^{*-Ns}}{L(2w-2s+1,\overline{\chi^*})} \sum_{n,m=1}^{\infty} \sum_{\substack{d_0d_1\cdots d_{N-1}=m\\ d_0|n,d_1|q_1,\dots,d_{N-2}|q_{N-2}}} \frac{\overline{\chi^*}(d_0d_1\dots d_{N-1})A\left(\tfrac{q_{N-2}d_{N-1}}{d_{N-2}},\dots,\tfrac{q_1d_2}{d_1},\tfrac{nd_1}{d_0}\right)}{(n/d_0)^{2w-s}d_0^{1+2w-2s}(d_1\dots d_{N-1})^{1-s}},
\end{aligned}$$

where we have combined the Fourier coefficients by the Hecke relation (4).We change the variable $n/d_0 \to n$. Then the sum over d_0 cancels with $L(2w-2s+1,\overline{\chi^*})$ in the denominator, giving

$$\begin{aligned}
Z(s,w) &= \frac{G(s)\tau(\chi^*)^N c^{*-Ns}}{L(2w-2s+1,\overline{\chi^*})} \sum_{n,d_0,d_{N-1}=1}^{\infty} \sum_{\substack{d_i|q_i\\ i=1,\dots,N-2}} \frac{\overline{\chi^*}(d_0d_1\dots d_{N-1})A\left(\tfrac{q_{N-2}d_{N-1}}{d_{N-2}},\dots,\tfrac{q_1d_2}{d_1},d_1n\right)}{n^{2w-s}d_0^{1+2w-2s}(d_1\dots d_{N-1})^{1-s}} \\
&= \frac{G(s)\tau(\chi^*)^N}{c^{*Ns}} \sum_{n,d_{N-1}=1}^{\infty} \sum_{\substack{d_i|q_i\\ i=1,\dots,N-2}} \frac{\overline{\chi^*}(d_1\dots d_{N-1})A\left(\tfrac{q_{N-2}d_{N-1}}{d_{N-2}},\dots,\tfrac{q_1d_2}{d_1},d_1n\right)}{n^{2w-s}(d_1\dots d_{N-1})^{1-s}}.
\end{aligned} \tag{30}$$

If we denote the right hand side of (27) by $\tau(\overline{\chi^*})b_n(s)$, our goal is to transform (30) into $R := \sum_{n=1}^{\infty} b_n(s)n^{-2w}$. But at this point it is easier to start from R. More explicitly, we have

$$R = \tau(\overline{\chi^*})^{-1}\sum_{h=1}^{\infty}\frac{1}{h^{2w}}\sum_{d_1|q_1,\ldots,d_{N-2}|q_{N-2}}\frac{\chi^*(d_1\cdots d_{N-2})}{(d_1\cdots d_{N-2})^s}\sum_{d\ell=h}\chi^*(d)G(\mathbf{q}',\ell c^*,\chi^*,s). \tag{31}$$

Here $\mathbf{q}'$ has been defined in (28). We plug in the definition of $G(\mathbf{q}',\ell c^*,\chi^*,s)$ from (10) for $\mathbf{q}'$, giving

$$\begin{aligned} G(\mathbf{q}',\ell c^*,\chi^*,s) = {} & \frac{G(s)}{c^{*Ns-1}\ell^{(N-2)s}}\sum_{f_1|\frac{q_1 d\ell}{d_1}}\sum_{f_2|\frac{q_1q_2d\ell}{f_1d_2}}\cdots\sum_{f_{N-2}|\frac{q_1\cdots q_{N-2}d\ell}{f_1\cdots f_{N-3}d_{N-2}}} \\ & \times\sum_{n=1}^{\infty}\frac{A(n,f_{N-2},\ldots,f_1)}{n^{1-s}f_1f_2\cdots f_{N-2}}\frac{f_1^{(N-1)s}f_2^{(N-2)s}\cdots f_{N-2}^{2s}}{q_1^{(N-2)s}q_2^{(N-3)s}\cdots q_{N-2}^{s}}\frac{(d_1\ldots d_{N-2})^s}{d^{(N-2)s}} \\ & \times g(\chi^*,\ell c^*,f_1)g(\chi^*,\tfrac{q_1d\ell c^*}{f_1d_1},f_2)\cdots g(\chi^*,\tfrac{q_1\cdots q_{N-3}d\ell c^*}{f_1\cdots f_{N-3}d_{N-3}},f_{N-2})g(\chi^*,\tfrac{q_1\cdots q_{N-2}d\ell c^*}{f_1\cdots f_{N-2}d_{N-2}},n). \end{aligned}$$

We substitute $G(\mathbf{q}',\ell c^*,\chi^*,s)$ with this expression in (31) and change the orders of summation between f_i and d_i. The summations over d_i collapse with the repeated use of Lemma 2.5, giving

$$\begin{aligned} R = {} & \tau(\overline{\chi^*})^{-1}\frac{G(s)}{c^{*Ns-1}}\sum_{h=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\substack{h|f_1\\ f_1|q_1h}}\sum_{\substack{\frac{q_1h}{f_1}|f_2\\ f_2|\frac{q_1q_2h}{f_1}}}\cdots\sum_{\substack{\frac{q_1\cdots q_{N-3}h}{f_1\cdots f_{N-3}}|f_{N-2}\\ f_{N-2}|\frac{q_1\cdots q_{N-2}h}{f_1\cdots f_{N-3}}}}\sum_{\frac{q_1\cdots q_{N-2}h}{f_1\cdots f_{N-2}}|n} \\ & \frac{\tau(\chi^*)^{N-1}}{h^{2w}}\overline{\chi^*}\left(\frac{f_1}{h}\right)\overline{\chi^*}\left(\frac{f_1f_2}{hq_1}\right)\ldots\overline{\chi^*}\left(\frac{f_1f_2\cdots f_{N-2}}{hq_1\cdots q_{N-3}}\right)\overline{\chi^*}\left(\frac{f_1f_2\cdots f_{N-2}n}{hq_1\cdots q_{N-2}}\right) \\ & \times\left(\frac{q_1}{f_1}\right)^{N-2}\left(\frac{q_2}{f_2}\right)^{N-3}\cdots\left(\frac{q_{N-2}}{f_{N-2}}\right)h^{N-1-Ns+2s}\frac{A(n,f_{N-2},\ldots,f_1)}{n^{1-s}f_1\cdots f_{N-2}}\frac{f_1^{(N-1)s}\cdots f_{N-2}^{2s}}{q_1^{(N-2)s}\cdots q_{N-2}^{s}}. \end{aligned}$$

Define the variables $e_1 = f_1/h$ and $e_i = f_1\cdots f_i/q_1\cdots q_{i-1}h$ for $i = 2,\ldots N-2$. The double conditions under the sums simplify to $e_i|q_i$. Also define $e_{N-1} = f_1\cdots f_{N-2}n/hq_1\cdots q_{N-2}$ and it runs over all positive integers. Finally noting $\tau(\overline{\chi^*})^{-1} = \tau(\chi^*)/c^*$, we get

$$R = \frac{G(s)\tau(\chi^*)^N}{c^{*Ns}}\sum_{h,e_{N-1}=1}^{\infty}\frac{1}{h^{2w-s}}\sum_{\substack{e_i|q_i\\ i=1,\ldots,N-2}}\frac{\overline{\chi^*}(e_1\cdots e_{N-2}e_{N-1})}{(e_1\cdots e_{N-1})^{1-s}}A(\tfrac{e_{N-1}q_{N-2}}{e_{N-2}},\ldots,\tfrac{e_2q_1}{e_1},e_1h),$$

which in turn, by (30), equals $Z(s,w)$ as well as (29). We complete the proof after applying the uniqueness theorem for Dirichlet series (Theorem 11.3 of [Apo76]) to the equality between (29) and (31). □

Remark 5.2. The above proof works for $N \geq 3$ but not for $N = 2$. We can prove the Voronoi formula for $\mathrm{SL}(2,\mathbb{Z})$ similarly and easily by considering

$$Z(s,w) = \frac{L(2w-s,F)L(s,F\times\chi^*)}{L(2w-2s+1,\overline{\chi^*})L(2w,\chi^*)}.$$

We have form the Hecke relation

$$Z(s,w) = \tau(\overline{\chi^*})^{-1}\sum_{\ell=1}^{\infty}\sum_{n=1}^{\infty}\frac{A(n)}{n^s}\frac{g(\overline{\chi^*},\ell c^*,n)}{\ell^{1+2w-2s}},$$

and applying the functional equation for $L(s,F\times\chi^*)$ we have

$$Z(s,w) = \tau(\chi^*)c^{*-2s}G(s)\sum_{\ell=1}^{\infty}\sum_{n=1}^{\infty}\frac{A(n)}{n^{1-s}}\frac{g(\chi^*,\ell c^*,n)}{\ell^{2w}}.$$

Applying the uniqueness theorem for Dirichlet series to the variable w, we get the Voronoi formula with Gauss sums on GL(2).

5.2 Proof of Theorem 1.2

We are going to obtain the equality (11) from Theorem 5.1 by a Möbius inversion technique.

Proof of Theorem 1.2. Define for $\mathbf{q} = (q_1, \ldots, q_{N-2})$

$$\mathcal{H}(\mathbf{q}; n) := \sum_{d_1|q_1,\ldots,d_{N-2}|q_{N-2}} \frac{\chi^*(d_1 \cdots d_{N-2})}{(d_1 \cdots d_{N-2})^s} \sum_{d\ell=n} \chi^*(d) H(\mathbf{q}', \ell c^*, \chi^*, s)$$

and

$$\mathcal{G}(\mathbf{q}; n) := \sum_{d_1|q_1,\ldots,d_{N-2}|q_{N-2}} \frac{\chi^*(d_1 \cdots d_{N-2})}{(d_1 \cdots d_{N-2})^s} \sum_{d\ell=n} \chi^*(d) G(\mathbf{q}', \ell c^*, \chi^*, s),$$

where we had denoted for abbreviation $\mathbf{q}' = (\frac{q_1 d}{d_1}, \frac{q_2 d_1}{d_2}, \ldots, \frac{q_{N-2} d_{N-3}}{d_{N-2}})$. Construct the following summation

$$\begin{aligned} T &:= \sum_{e_0|n} \sum_{e_1|q_1 e_0} \cdots \sum_{e_{N-2}|q_{N-2}e_{N-3}} \frac{\mu(e_0 \cdots e_{N-2}) \chi^*(e_0 \cdots e_{N-2})}{(e_1 \cdots e_{N-2})^s} \mathcal{H}\left(\frac{q_1 e_0}{e_1}, \ldots, \frac{q_{N-2} e_{N-3}}{e_{N-2}}; \frac{n}{e_0} \right) \\ &= \sum_{e_0|n} \sum_{e_1|q_1 e_0} \cdots \sum_{e_{N-2}|q_{N-2}e_{N-3}} \frac{\mu(e_0 \cdots e_{N-2}) \chi^*(e_0 \cdots e_{N-2})}{(e_1 \cdots e_{N-2})^s} \sum_{\substack{d_i | q_i e_{i-1}/e_i \\ i=1,\ldots,N-2}} \frac{\chi^*(d_1 \cdots d_{N-2})}{(d_1 \cdots d_{N-2})^s} \\ &\qquad \times \sum_{d_0|n/e_0} \chi^*(d_0) H\left(\frac{q_1 e_0 d_0}{e_1 d_1}, \ldots, \frac{q_{N-2} e_{N-3} d_{N-3}}{e_{N-2} d_{N-2}}, \ \frac{n}{e_0 d_0} c^*, \chi^*, s \right). \end{aligned}$$

Change variables $e_i d_i \to a_i$ for $i = 0, \ldots, N-2$, and change orders of summation, getting

$$\begin{aligned} T &= \sum_{a_0|n} \sum_{e_0|a_0} \sum_{a_1|q_1 e_0} \sum_{e_1|a_1} \cdots \sum_{a_{N-2}|q_{N-2}e_{N-3}} \sum_{e_{N-2}|a_{N-2}} \frac{\chi^*(a_0 \cdots a_{N-2})}{(a_1 \cdots a_{N-2})^s} \\ &\qquad \times H\left(\frac{q_1 a_0}{a_1}, \frac{q_2 a_1}{a_2}, \ldots, \frac{q_{N-2} a_{N-3}}{a_{N-2}}, \ \frac{nc^*}{a_0}, \chi^*, s \right) \mu(e_0) \cdots \mu(e_{N-2}). \end{aligned}$$

One by one the the Möbius summation over e_i will force $a_i = 1$, and we obtain $T = H(\mathbf{q}, nc^*, \chi^*, s)$. By Theorem 5.1, we have $\mathcal{H} = \mathcal{G}$ and and the same calculations yield $T = G(\mathbf{q}, nc^*, \chi^*, s)$. This proves the theorem. $\square$

5.3 Equivalence to the Average Voronoi Formula

First we prove a lemma showing that the hyper-Kloosterman sum on the right hand side of (8) becomes a product of $(N-2)$ Gauss sums after averaging against a Dirichlet character.

Lemma 5.3. *Let χ be a Dirichlet character modulo c which is induced from the primitive character χ^* modulo c^*. Let $\mathbf{q} = (q_1, \ldots, q_{N-2})$ and $\mathbf{d} = (d_1, \ldots, d_{N-2})$ be two tuples of positive integers, and assume that all the divisibility conditions in* (1) *are met. Consider the summation*

$$S := \sum_{\substack{a \bmod c \\ (a,c)=1}} \chi(a) \operatorname{Kl}(a, n, c; \mathbf{q}, \mathbf{d}).$$

The quantity S is zero unless the divisibility conditions

$$d_1 c^* | q_1 c, \quad d_2 c^* \left| \frac{q_1 q_2 c}{d_1}, \right. \quad d_3 c^* \left| \frac{q_1 q_2 q_3 c}{d_1 d_2}, \right. \quad \ldots, \quad d_{N-2} c^* \left| \frac{q_1 \cdots q_{N-2} c}{d_1 \cdots d_{N-3}}, \right. \tag{32}$$

are satisfied. Under such divisibility conditions, S can be written as a product of Gauss sums

$$S = g(\chi^*, c, d_1) g(\chi^*, \tfrac{q_1 c}{d_1}, d_2) \cdots g(\chi^*, \tfrac{q_1 \cdots q_{N-3} c}{d_1 \cdots d_{N-3}}, d_{N-2}) g(\chi^*, \tfrac{q_1 \cdots q_{N-2} c}{d_1 \cdots d_{N-2}}, n).$$

Proof. The divisibility conditions (1) imply

$$d_1|q_1(c,d_1),\quad d_2\left|q_2\left(\frac{q_1c}{d_1},d_2\right),\quad d_3\right|q_3\left(\frac{q_1q_2c}{d_1d_2},d_3\right),\ldots,d_{N-2}\left|q_{N-2}\left(\frac{q_1\ldots q_{N-3}c}{d_1\ldots d_{N-3}},d_{N-2}\right).\right. \tag{33}$$

We open up the hyper-Kloosterman sum in S. Our forthcoming computation is an iterative process. The summation over a yields a Gauss sum, which in turn produces the term $\overline{\chi^*}(x_1)$. Then the summation over x_1 yields another Gauss sum, which produces the term $\overline{\chi^*}(x_2)$ and so on.

Firstly we sum over a modulo c

$$\begin{aligned}S&=\sum_{a\mod c}\chi(a)\sideset{}{^*}\sum_{x_1(\mathrm{mod}\ \frac{q_1c}{d_1})}e\left(\frac{d_1x_1a}{c}\right)\left(\sideset{}{^*}\sum_{x_2(\mathrm{mod}\ \frac{q_1q_2c}{d_1d_2})}e\left(\frac{d_2x_2\overline{x_1}}{\frac{q_1c}{d_1}}\right)\cdots\right)\\&=\sideset{}{^*}\sum_{x_1(\mathrm{mod}\ \frac{q_1c}{d_1})}g(\chi^*,c,x_1d_1)\left(\sideset{}{^*}\sum_{x_2(\mathrm{mod}\ \frac{q_1q_2c}{d_1d_2})}e\left(\frac{d_2x_2\overline{x_1}}{\frac{q_1c}{d_1}}\right)\cdots\right).\end{aligned}$$

Now because $(c,x_1d_1)=((c,q_1c),x_1d_1)=(c,(q_1c,x_1d_1))=(c,d_1)$, we deduce from Lemma 2.3 that

$$g(\chi^*,c,x_1d_1)=\overline{\chi^*(x_1)}g(\chi^*,c,d_1).$$

By Lemma 2.3, this Gauss sum is zero unless $c^*|\frac{c}{(c,d_1)}$, which implies the first divisibility condition of (32) because $c^*|\frac{c}{(c,d_1)}=\frac{d_1}{(c,d_1)}\frac{c}{d_1}|\frac{q_1c}{d_1}$ by (33).

Next we sum over x_1. Notice that $\overline{x_1}$ is its multiplicative inverse modulo q_1c/d_1 and hence modulo c^*. This means that $\chi^*(\overline{x_1})=\overline{\chi^*(x_1)}$. We change variables in the x_1 summation $x_1\to\overline{x_1}$, and change orders of summation to obtain

$$\begin{aligned}S&=g(\chi^*,c,d_1)\sideset{}{^*}\sum_{x_1(\mathrm{mod}\ \frac{q_1c}{d_1})}\overline{\chi^*(x_1)}\left(\sideset{}{^*}\sum_{x_2(\mathrm{mod}\ \frac{q_1q_2c}{d_1d_2})}e\left(\tfrac{d_2x_2\overline{x_1}}{\frac{q_1c}{d_1}}\right)\cdots\right)\\&=g(\chi^*,c,d_1)\sideset{}{^*}\sum_{x_2(\mathrm{mod}\ \frac{q_1q_2c}{d_1d_2})}\ \sideset{}{^*}\sum_{x_1(\mathrm{mod}\ \frac{q_1c}{d_1})}\chi^*(x_1)e\left(\tfrac{d_2x_2x_1}{\frac{q_1c}{d_1}}\right)\left(\sideset{}{^*}\sum_{x_3(\mathrm{mod}\ \frac{q_1q_2q_3c}{d_1d_2d_3})}e\left(\tfrac{d_3x_3\overline{x_2}}{\frac{q_1q_2c}{d_1d_2}}\right)\cdots\right)\\&=g(\chi^*,c,d_1)\sideset{}{^*}\sum_{x_2(\mathrm{mod}\ \frac{q_1q_2c}{d_1d_2})}g(\chi^*,\tfrac{q_1c}{d_1},d_2x_2)\left(\sideset{}{^*}\sum_{x_3(\mathrm{mod}\ \frac{q_1q_2q_3c}{d_1d_2d_3})}e\left(\tfrac{d_3x_3\overline{x_2}}{\frac{q_1q_2c}{d_1d_2}}\right)\cdots\right).\end{aligned}$$

Once again the equalities $(\frac{q_1c}{d_1},d_2x_2)=((\frac{q_1c}{d_1},\frac{q_1q_2c}{d_1}),d_2x_2)=(\frac{q_1c}{d_1},(\frac{q_1q_2c}{d_1},d_2x_2))=(\frac{q_1c}{d_1},d_2)$ imply that we can pull out $\overline{\chi^*}(x_2)$ from the Gauss sum. Then we have

$$S=g(\chi^*,c,d_1)g(\chi^*,\tfrac{q_1c}{d_1},d_2)\sideset{}{^*}\sum_{x_2(\mathrm{mod}\ \frac{q_1q_2c}{d_1d_2})}\overline{\chi^*(x_2)}\left(\sideset{}{^*}\sum_{x_3(\mathrm{mod}\ \frac{q_1q_2q_3c}{d_1d_2d_3})}e\left(\tfrac{d_3x_3\overline{x_2}}{\frac{q_1q_2c}{d_1d_2}}\right)\cdots\right).$$

The second Gauss sum $g(\chi^*,\frac{q_1c}{d_1},d_2)$ vanishes unless $c^*|\frac{q_1c/d_1}{(q_1c/d_1,d_2)}$ by Lemma 2.3. This in turn implies $c^*|\frac{q_1c/d_1}{(q_1c/d_1,d_2)}|\frac{cq_1q_2}{d_1d_2}$ by (33), which is the second divisibility condition of (32). We complete the proof after repeating this process $(N-2)$ times. □

Proposition 5.4. *Theorem 1.2 and Theorem 1.3 are equivalent.*

Proof. Multiply both sides of (8) by $\chi(a)$ and sum over reduced residue classes modulo c. On the left hand side of (8), one gets

$$\sum_{\substack{a\mod c\\(a,c)=1}}\chi(a)L_{\mathbf{q}}(s,F,a/c)=(c/c^*)^{1-2s}H(\mathbf{q},c,\chi^*,s),$$

whereas on the right hand side of (8), one obtains $(c/c^*)^{1-2s}G(\mathbf{q}, c, \chi^*, s)$ by making use of Lemma 5.3 and the fact that

$$g(\chi^*, \tfrac{q_1\cdots q_{N-2}c}{d_1\cdots d_{N-2}}, -n) = \pm g(\chi^*, \tfrac{q_1\cdots q_{N-2}c}{d_1\cdots d_{N-2}}, n),$$

depending on whether $\chi(-1)$ is 1 or -1. This shows that Theorem 1.3 implies Theorem 1.2.

Conversely if we multiply both sides of the equality $H(\mathbf{q}, c, \chi^*, s) = G(\mathbf{q}, c, \chi^*, s)$ of Theorem 1.2 by $\frac{1}{\phi(c)}\overline{\chi(a)}$ and sum over all Dirichlet characters (both primitive and non-primitive) modulo c, we obtain Theorem 1.3, by using the orthogonality relation for Dirichlet characters.

Since both of the aforementioned summations that shuttle between Theorem 1.3 and Theorem 1.2 are finite, the properties of analytic continuation are preserved. □

Acknowledgement

The authors would like to thank Dorian Goldfeld, Wenzhi Luo for helpful suggestions, Matthew Young for his extensive help in organization of the manuscript and his encouragement, and Jeffrey Hoffstein, in whose class the seed for this work originated.

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